Examples for linear programming

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1. Use simplex method.

$$\begin{aligned} \text{maximise:} \quad & z = x_1 + 9x_2 + x_3 \\ \text{subject to:} \quad & x_1 + 2x_2 + 3x_3 \leq 9 \\ & 3x_1 + 2x_2 + 2x_3 \leq 15 \\ \text{with:} \quad & \text{all variables non-negative} \end{aligned}$$

Solution.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 9 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 3 & 2 & 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 15 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$$

cf.

$$\begin{aligned} \text{optimise:} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} = \mathbf{b} \\ \text{with:} \quad & \mathbf{x} \geq 0 \end{aligned}$$

		$\mathbf{x}^T$ $\mathbf{c}^T$	
$\mathbf{x}_0$	$\mathbf{c}_0$	A	$\mathbf{b}$
		$\pm\left(\mathbf{c}^T - \mathbf{c}_0^T A\right)$	$\mp\mathbf{c}_0^T\mathbf{b}$

Note: $\left(\mathbf{c}^T - \mathbf{c}_0^T A\right)$ and $-\mathbf{c}_0^T\mathbf{b}$ in case of a minimisation problem, whereas $-\left(\mathbf{c}^T - \mathbf{c}_0^T A\right)$ and $\mathbf{c}_0^T\mathbf{b}$ in case of a maximisation problem.

Tableau 1;

	x_1	x_2	x_3	x_4	x_5	
x_4	1	2	3	1	0	9
x_5	3	2	2	0	1	15
	-1	-9	-1	0	0	0

The most negative number in the last row is -9. Therefore x_2-column becomes the work column. And then,

$$\begin{array}{ccccccc} & x_2 & & & & \\ x_4 & 2 & \rightarrow & \text{positive} & \rightarrow & \frac{9}{2} = 4.5 \\ x_5 & 2 & \rightarrow & \text{positive} & \rightarrow & \frac{15}{2} = 7.5 \end{array}$$

Since $\min(4.5, 7.5) = 4.5$, the value of x_2 on the row corresponding to x_4 becomes our pivot element. Then carry out a series of elementary row operations, namely in order $(I)_2 \leftarrow (I)_1/2$; $(II)_2 \leftarrow (II)_1 - 2(I)_2$; $(III)_2 \leftarrow (III)_1 + 9(I)_2$;

	x_1	x_2	x_3	x_4	x_5	
x_2	$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{9}{2}$
x_5	2	0	-1	-1	1	6
	$\frac{7}{2}$	0	$\frac{25}{2}$	$\frac{9}{2}$	0	$\frac{81}{2}$

Now the last row is all non-negative. Therefore $x_2^* = \frac{9}{2}$, $x_5^* = 6$, $x_1^* = x_3^* = x_4^* = 0$ and $z^* = \frac{81}{2}$

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